Shape Classification of Parametric Cubic Segments

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1 Introduction

Polynomial cubic and rational cubic curves have been widely used in computer-aided design. However, the polynomial cubic curves do not always generate "visually pleasing", "shape preserving" (or simply "fair") interpolants which do not contain *unwanted* inner inflection points and singularities to a set of planar data points. There is a considerable literature on numerical methods for generating shape preserving interpolation; see for example, [11], [12], and the references therein. A way to overcome this problem is to consider the rational cubic curve segments $z(t), 0 \leq t \leq 1, u = 1 - t$ with a *single* rationality parameter $p > 0$, for example,

$$z(t) \in Span\{t, u, t^2u/(1 + ptu), tu^2/(1 + ptu)\} \tag{1}$$

and

$$z(t) \in Span\{t, u, t^3/(1 + pu), u^3/(1 + pt)\}. \tag{2}$$

In [6] - [7], we examined the distribution of inflection points and singularities on the rational cubic curve segments.

In Sections 2-3, we consider the distribution of inflection points and singularities (a loop and a cusp) on the rational cubic curve of the form with *two* positive parameters $w_i, i = 1, 2$:

$$cubic/\{u^3 + w_1u^2t + w_2ut^2 + t^3\}, \qquad u = 1 - t, 0 \leq t \leq 1. \tag{3}$$

For given planar data $z_j^{(k)}, j = 0, 1; k = 0, 1$, the rational curve segment of the form (1.3) satisfying $z^{(k)}(j) = z_j^{(k)}, j = 0, 1; k = 0, 1$ is given by

$$z(t) = \frac{u^3z_0 + u^2t(z_0' + w_1z_0) + ut^2(-z_1' + w_2z_1) + t^3z_1}{u^3 + w_1u^2t + w_2ut^2 + t^3}. \tag{4}$$

The rational cubic curve segment (1.3)(or (1.4)) has more flexibility than the above rational curve segments (1.1) and (1.2) since it has two degrees of freedom. Note that the rational cubic curve segment (1.1) is a special case of (1.3) with $w_1 = w_2 = 3 + p$ and that $w_1 = w_2 = 3$ reduces the curve segment (1.3) to the well-known polynomial cubic one. In what follows, given two vectors $A = (A_1, A_2), B = (B_1, B_2)$, we write $A \times B = A_1B_2 - A_2B_1$. Note that if $z_i', i = 1, 2$ are not parallel, i.e., $z_0' \times z_1' \neq 0$, then any vector, for example, $\Delta z (= z_1 - z_0)$ can be represented as follows: $\Delta z = \lambda z_0' + \mu z_1'$ where λ and μ can be solved in terms of $z_i', i = 0, 1$ and Δz as

$$(z_0' \times z_1')(\lambda, \mu) = (-z_1' \times \Delta z, z_0' \times \Delta z) \tag{5}$$

The object of Sections 2-3 gives the distribution of the inflection points and singularities on the curve of the form (1.3) with respect to (λ, μ). It shows that the curve segment of the form (1.3) is fair for $\lambda \geq 1/w_1, \mu \geq 1/w_2$. Section 4 gives an application of the distribution to the shape determination of the rational cubic Bézier curve segment resulting from placing one of the four control points in various regions with the remaining three control points fixed.

2 Inflection points and singularities on rational cubic curve segments (1.4)

We state the main Theorem 2.1 and Fig.1 concerning the distribution of inflection points and singularity on the parametric rational cubic segment (1.4). Let the curve $k_1(\lambda, \mu) = 0$ be a branch of $k(\lambda, \mu) = 0$ represented by

$$(\lambda, \mu) = (\frac{-t^4 + w_2 t^2 + 2t}{3t^2 + 2w_1 t + 2w_2 t^3 + w_1 w_2 t^2}, \frac{2t^3 + w_1 t^2 - 1}{3t^2 + 2w_1 t + 2w_2 t^3 + w_1 w_2 t^2}), \qquad t > 0 \qquad (6)$$

where

$$\begin{aligned} k(\lambda, \mu) &= 4\lambda^3(w_2\mu - 1) + 4\mu^3(w_1\lambda - 1) - 3\lambda^2\mu^2 \\ &+ (w_1\lambda - 1)^2(w_2\mu - 1)^2 - 6\lambda\mu(w_1\lambda - 1)(w_2\mu - 1). \end{aligned} \qquad (7)$$

The branch $k_1(\lambda, \mu) = 0$ lies in the region limited by $\lambda < 1/w_1, \mu < 1/w_2$ and $\mu^2 = \lambda(w_2\mu - 1)$, $\lambda^2 = \mu(w_1\lambda - 1)$. *Mathematica* (A System for Doing Mathematics by Computer) greatly helps us check that $k_1 = 0$ is one branch of $k = 0$. It has two straight lines $w_1\lambda = 1, w_2\mu = 1$ as its asymptotic lines. Let the symbols A, B, and C denote the branches of the hyperbolas: $\mu^2 = \lambda(w_2\mu - 1)$, $\lambda^2 = \mu(w_1\lambda - 1)$, and $k_1(\lambda, \mu) = 0$, respectively. Then the following main theorem 2.1 provides a scheme for the adjustment which plays an important role in shape control.

Theorem 2.1 Assume that $\Delta z = \lambda z_0' + \mu z_1'$ with $z_0' \times z_1' \neq 0$. Then, Fig. 1 gives the distribution of inflections and singularity on the curve of the form (1.4) with respect to (λ, μ) where (i) $N_i, 0 \leq i \leq 2$ represent the regions for which the curve has i-inflection points and no singularity, (ii) C (or L limited by A, B, C) means the region for the curve to have a cusp (or a loop) and no inflection point. Precisely speaking of the boundaries of the regions, N_0 contains its all boundaries including A, B, and N_1 contains the two straight lines: $\lambda = 1/w_1, \mu < 1/w_2$ and $\lambda < 1/w_1, \mu = 1/w_2$.

When $w_i = 3, i = 1, 2$ (i.e., the polynomial cubic case), $k_1(\lambda, \mu) = 0$ reduces to a branch of the hyperbola: $(\lambda - 1/3)(\mu - 1/3) = 1/36$ limited by $\lambda, \mu < 1/3$.

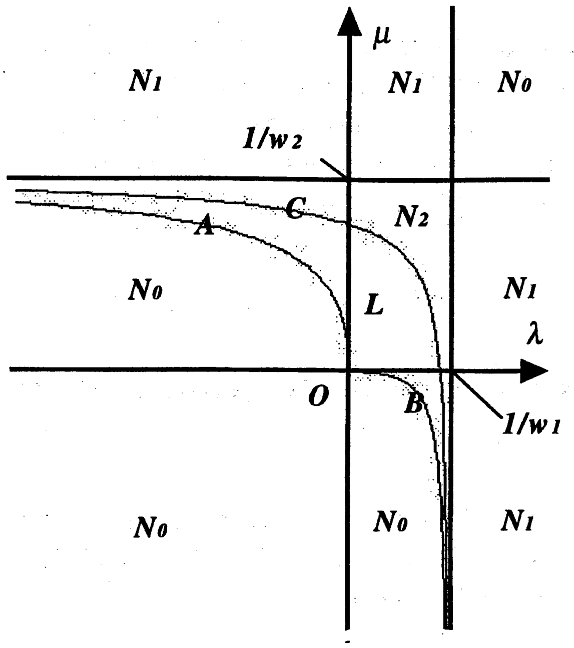

Fig. 1. Distribution of inflections and singularity.

3 Proof of Theorem 2.1

From (1.4), we obtain

$$z'(t) = \frac{a(t)z_0' + b(t)z_1'}{(u^3 + w_1u^2t + w_2ut^2 + t^3)^2}, \qquad u = 1 - t \tag{8}$$

where

$$\begin{aligned} a(t) &= u(u^3 - 2t^3 - w_2t^2u) + \lambda tu(2t^2w_1 + 2u^2w_2 + 3tu + tuw_1w_2) \\ b(t) &= t(t^3 - 2u^3 - w_1u^2t) + \mu tu(2t^2w_1 + 2u^2w_2 + 3tu + tuw_1w_2). \end{aligned} \tag{9}$$

Inflection points: Inflection points are determined by $z'(t) \times z''(t) = 0, 0 < t < 1$ or equivalently, $a'(t)b(t) - a(t)b'(t) = 0$. Letting

$$\begin{aligned} \tau(t) &= t^2(3u + w_1t)(1 - 3tu + w_1tu^2 + w_2t^2u) \\ \theta(t) &= u^2(3t + w_2u)(1 - 3tu + w_1tu^2 + w_2t^2u) \\ \rho(t) &= (1 - 3tu)(1 - 3tu + w_1tu^2 + w_2t^2u), \end{aligned} \tag{10}$$

the inflection points are determined by

$$\lambda\tau(t) + \mu\theta(t) = \rho(t), u = 1 - t, 0 < t < 1. \tag{11}$$

Substitution of $t = 1/(1+\sigma)$ changes the above equation (3.4) into a product of two *cubic* polynomials:

$$(\sigma^3 + w_1\sigma^2 + w_2\sigma + 1)\{(\mu w_2 - 1)\sigma^3 + 3\mu\sigma^2 + 3\lambda\sigma + (\lambda w_1 - 1)\} = 0, \sigma > 0. \tag{12}$$

Since $w_1, w_2 > 0$, we obtain a *cubic* equation:

$$(\mu w_2 - 1)\sigma^3 + 3\mu\sigma^2 + 3\lambda\sigma + (\lambda w_1 - 1) = 0. \tag{13}$$

The number of the inflection points being equal to the number of the positive roots of the above *cubic* equation, easily we have

(a) $\lambda \geq 1/w_1, \mu \geq 1/w_2$: $(\lambda, \mu) \in N_0$.

(b) $(\lambda - 1/w_1)(\mu - 1/w_2) < 0, \lambda = 1/w_1, \mu < 1/w_2, \lambda < 1/w_1, \mu = 1/w_2$: $(\lambda, \mu) \in N_1$.

Next consider the remaining case.

(c) $\lambda < 1/w_1, \mu < 1/w_2$: Then we rewrite equation (3.6) as

$$\lambda = g(\sigma)(= \frac{(1 - w_2\mu)\sigma^3 - 3\mu\sigma^2 + 1}{3\sigma + w_1}), \qquad \sigma > 0 \tag{14}$$

where

$$(1/3)(3\sigma + w_1)^2 g'(\sigma) = 2(1 - w_2\mu)\sigma^3 + \{w_1(1 - w_2\mu) - 3\mu\}\sigma^2 - 2w_1\mu\sigma - 1. \tag{15}$$

Since $g'(0) < 0$ and $g'(+\infty) > 0$, Decartes' Rule of Signs (the number of the positive roots of a polynomial being no greater than that of sign changes of its coefficient sequence) gives g' has only one positive zero point, say q where the sequence is given by (i): for $\mu \leq 0$, $(+, +, ?, -)$ and (ii): for $\mu > 0$, $(+, ?, -, -)$. Then, $g(q)$ being its minimum over $(0, \infty)$, $(\lambda, \mu) \in N_2$ or N_0 for $g(q) < \lambda < 1/w_1$ or $\lambda \leq g(q)$, respectively.

Here, note

$$\mu = \frac{2q^3 + w_1q^2 - 1}{3q^2 + 2w_1q + 2w_2q^3 + w_1w_2q^2}, \qquad g(q) = \frac{(1 - w_2\mu)q^3 - 3\mu q^2 + 1}{3q + w_1}. \tag{16}$$

Use (3.9) to get

$$g(q) = \frac{-q^4 + w_2q^2 + 2q}{3q^2 + 2w_1q + 2w_2q^3 + w_1w_2q^2} \tag{17}$$

from which $(g(q), \mu)$ is on the branch $k_1(\lambda, \mu) = 0$ (note (2.1)). Hence we have

Lemma If $(\lambda, \mu) \in N_i, 0 \leq i \leq 2$, the curve (1.4) has i-inflection points where $N_0 = \{(\lambda, \mu) | \ \lambda \geq 1/w_1, \mu \geq 1/w_2)$ or $k_1(\lambda, \mu) \geq 0\}$, $N_1 = \{(\lambda, \mu) | \ (\lambda - 1/w_1)(\mu - 1/w_2) \leq 0$ or $\lambda = 1/w_1, \mu < 1/w_2$ or $\lambda < 1/w_1, \mu = 1/w_2\}$ and $N_2 = \{(\lambda, \mu) | \ k_1(\lambda, \mu) < 0, \lambda < 1/w_1, \mu < 1/w_2\}$.

Singularities: A loop occurs if $z(\alpha) = z(\beta)$ for $0 < \alpha \neq \beta < 1$. Since z_0' and z_1' are independent, letting the coefficients of the two vectors in $\{z(\alpha) - z(\beta)\}$ be zero gives

$$\begin{aligned}
&\lambda\left[\{\beta^3 + w_2(1-\beta)\beta^2\}\varphi(\alpha) - \{\alpha^3 + w_2(1-\alpha)\alpha^2\}\varphi(\beta)\right] \\
&= (1-\alpha)^2\alpha\varphi(\beta) - (1-\beta)^2\beta\varphi(\alpha) \\
&\mu\left[\{\beta^3 + w_2(1-\beta)\beta^2\}\varphi(\alpha) - \{\alpha^3 + w_2(1-\alpha)\alpha^2\}\varphi(\beta)\right] \\
&= (1-\beta)\beta^2\varphi(\alpha) - (1-\alpha)\alpha^2\varphi(\beta)
\end{aligned} \tag{18}$$

where $\varphi(t)$ is the denominator of (1.4), i.e., $\varphi(t) = u^3 + w_1u^2t + w_2ut^2 + t^3$. Note $\alpha \neq \beta$ to obtain

$$\begin{aligned}
\lambda &= \{-(1-\alpha)^2(1-\beta)^2 + \alpha\beta(\alpha+\beta-2\alpha\beta) + w_2\alpha\beta(1-\alpha)(1-\beta)\}/D \\
\mu &= \{(1-\alpha)(1-\beta)(\alpha+\beta-2\alpha\beta) - \alpha^2\beta^2 + w_1\alpha\beta(1-\alpha)(1-\beta)\}/D
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
D = {} & \beta^2(1-\alpha)^2 + \alpha\beta(1-\alpha)(1-\beta) + \alpha^2(1-\beta)^2 + w_1\alpha\beta(\alpha+\beta-2\alpha\beta) \\
& + w_2(1-\alpha)(1-\beta)(\alpha+\beta-2\alpha\beta) + w_1w_2(1-\alpha)(1-\beta)\alpha\beta.
\end{aligned} \tag{20}$$

For easy check of the calculation, replace $(\alpha, \beta) = (1/(1+\sigma_1), 1/(1+\sigma_2))$ and in addition $(m, n) = (\sigma_1 + \sigma_2, \sigma_1\sigma_2)$ to get

$$\lambda = \frac{-n^2 + m + w_2n}{m^2 - n + w_1m + w_2mn + w_1w_2n}, \quad \mu = \frac{nm - 1 + w_1n}{m^2 - n + w_1m + w_2mn + w_1w_2n}. \tag{21}$$

Now, *Mathematica* again greatly helps us check that the above (λ, μ) satisfies

$$\begin{aligned}
(i) \quad & (m^2 - n + w_1m + w_2mn + w_1w_2n)^4 k_1(\lambda, \mu) \\
& = (m^2 - 4n)(1 + w_2m + w_1m^2 + m^3 - 2w_1n + w_2^2n - 3mn \\
& + w_1w_2mn + w_2m^2n + w_1^2n^2 - 2w_2n^2 + w_1mn^2 + n^3)^2 \\
(ii) \quad & \{\lambda^2 - \mu(w_1\lambda - 1)\} = n\{\mu^2 - \lambda(w_2\mu - 1)\} \\
(iii) \quad & (m^2 - n + w_1m + w_2mn + w_1w_2n)^2\{\mu^2 - \lambda(w_2\mu - 1)\} \\
& = m^3 + (w_1 + w_2n)m^2 + (w_2 - 3n + w_1w_2n + w_1n^2)m \\
& + 1 - 2w_1n + w_2^2n + w_1^2n^2 - 2w_2n^2 + n^3 (= r(m)).
\end{aligned} \tag{22}$$

Here we note

$$\begin{aligned}
(i) \quad & m^2 - 4n = (\sigma_1 - \sigma_2)^2 > 0 \text{ since } \alpha \neq \beta \leftrightarrow \sigma_1 \neq \sigma_2 \\
(ii) \quad & r'(m) > 0 \text{ on } [2\sqrt{n}, \infty), r(2\sqrt{n}) = (8 + 4w_2m + 2w_1m^2 + m^3)^2/64 > 0, m = 2\sqrt{n}.
\end{aligned} \tag{23}$$

Hence, $(\lambda, \mu) \in L$ if

$$k_1(\lambda, \mu) > 0, \lambda^2 > \mu(w_1\lambda - 1), \mu^2 > \lambda(w_2\mu - 1). \tag{24}$$

Conversely, if (λ, μ) satisfies the above inequalities (3.17), first note $m^2 - 4n > 0$ by 3.15(i). In addition,

$$\begin{aligned}
&(i) \quad n = \{\lambda^2 - \mu(w_1\lambda - 1)\}/\{\mu^2 - \lambda(w_2\mu - 1)\} \\
&(ii) \quad (n\lambda - \mu)^2\{\mu^2 - \lambda(w_2\mu - 1)\}^3 m = (1 - w_1\lambda - w_2\mu - \lambda\mu + w_1w_2\lambda\mu) \\
&\qquad \times(\lambda^3 - w_1\lambda^2\mu + w_2\lambda\mu^2 - \mu^3)^2.
\end{aligned} \tag{25}$$

Here it is easy to show that $1 - w_1\lambda - w_2\mu - \lambda\mu + w_1w_2\lambda\mu > 0$ or $(1 - w_1\lambda)(1 - w_2\mu) > \lambda\mu$ under (3.17) as follows. If $\lambda\mu \leq 0$, the inequality easily follows from $\lambda < 1/w_1, \mu < 1/w_2$. If $\lambda, \mu > 0$, then we only have to note that $k_1(\lambda, \mu) = 0 \Leftrightarrow 4\lambda^3(1 - w_2\mu) + 4\mu^3(1 - w_1\lambda) + 3\lambda^2\mu^2 = (w_1\lambda - 1)(w_2\mu - 1)\{(w_1\lambda - 1)(w_2\mu - 1) - 6\lambda\mu\} > 0 \Rightarrow (w_1\lambda - 1)(w_2\mu - 1) > 6\lambda\mu$. Hence, $n, m > 0, m^2 - 4n > 0 \to \sigma_1 \neq \sigma_2 > 0$ under (3.17), i.e., there exists (α, β) such that $z(\alpha) = z(\beta), 0 < \alpha \neq \beta < 1$ and so $(\lambda, \mu) \in L$.
A cusp of a curve can be regarded as the limit of a loop when $\sigma_1 = \sigma_2$, i.e., with $m = 2t > 0$, and so from (3.14)

$$\lambda = \frac{-t^4 + w_2t^2 + 2t}{3t^2 + 2w_1t + 2w_2t^3 + w_1w_2t^2}, \quad \mu = \frac{2t^3 + w_1t^2 - 1}{3t^2 + 2w_1t + 2w_2t^3 + w_1w_2t^2}. \tag{26}$$

Hence (λ, μ) is on the branch $k_1 = 0$ (note (2.1)), and so we obtain **Lemma 2** If $(\lambda, \mu) \in L$ or C, then a loop or a cusp occurs on the curve segment (1.4) where $L = \{(\lambda, \mu) | \; k_1(\lambda, \mu) > 0, \lambda^2 > \mu(w_1\lambda - 1), \; \mu^2 > \lambda(w_2\mu - 1) \; \}$ and $C = \{(\lambda, \mu) | \; k_1(\lambda, \mu) = 0\}$. Lemmas 3.1-3.2 give the desired Theorem 2.1 on the distribution of inflection points and singularities on the planar rational curves of the form (1.4) where the inflection points, cusps or loops do not occur simultaneously.

4 Shape classification of rational cubic Bézier curve

Letting $p_i, 1 \leq i \leq 4$ be control vertices belonging to R^2, then the rational cubic Bézier curve of the standard form with the weights $w_i/3, i = 1, 2$ is given by

$$z(t) = \frac{u^3p_0 + w_1u^2tp_1 + w_2ut^2p_2 + t^3p_3}{u^3 + w_1u^2t + w_2ut^2 + t^3}, \qquad u = 1 - t, 0 \leq t \leq 1 \tag{27}$$

or equivalently by (1.4) where

$$z_0 = p_0, z_0' = w_1(p_1 - p_0), z_1' = w_2(p_3 - p_2), z_1 = p_3. \tag{28}$$

The curve segment of the form (4.1) resulting from placing p_1 in various regions of the plane, with p_0, p_2, p_3 fixed, is considered. Then, $\Delta z = \lambda z_0' + \mu z_1'$ is equivalent to

$$p_3 - p_0 = \lambda w_1(p_1 - p_0) + \mu w_2(p_3 - p_2) \tag{29}$$

from which follows

$$p_1 - p_2 = u(p_0 - p_2) + v(p_3 - p_2), \qquad u = 1 - \frac{1}{\lambda w_1}, v = \frac{1 - \mu w_2}{\lambda w_1}. \tag{30}$$

Theorem 2.1 gives Fig.2 (the shape classification of the rational cubic Bézier curve for placement of p_1 with p_0, p_2, p_3 fixed) where for A, B, C in Theorem 2.1, (u, v) can be given respectively:

$$\begin{aligned} &A : u = 1 - v + \sqrt{w_2^2/w_1(-v)}, v < 0, B : v = 1 - u - w_2/(w_1^2 u), u < 0 \\ &C : u = \frac{t(w_1t^2 + 2w_2t + 3)}{w_1(t^3 - w_2t - 2)}, v = -\frac{(3t^2 + 2w_1t + w_2)}{w_1t(t^3 - w_2t - 2)}, \qquad t > 0 \end{aligned} \tag{31}$$

for A, B. Since C approaches to the straight line p_0p_2 as $w_i, i = 1, 2 \to \infty$, N_2 disappears. Hence, the curve (4.1) has at most one inflection point or a loop.

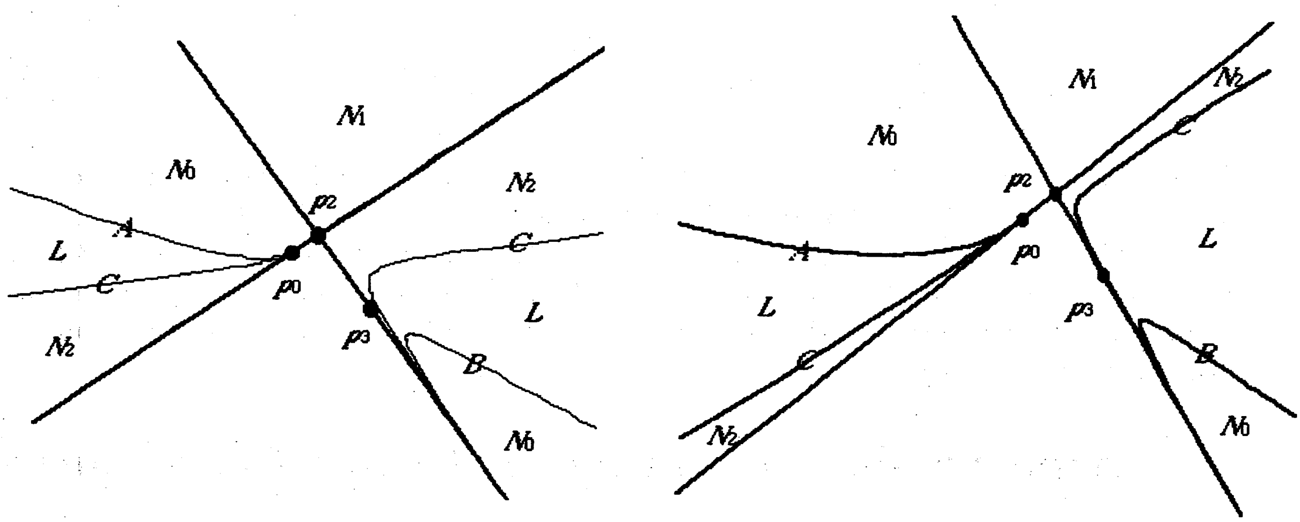

Fig. 2. Shape classification with $(w_1, w_2) = (4, 3)$ and $(16, 12)$

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